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On two sets of orthogonal polynomial systems encountered in non-linear physics

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Abstract. Two sets with an infinite number of new systems of orthogonal polynomials have recently been discovered by Smith in connection with some non-linear physical problems, e.g. the dispersion of a buoyant contaminant in a fluid. They appear as solutions of non-linear differential equations. Let $\{P_n(x; m, k, \delta)\}$ and $\{Q_n(x; m, k)\}$, with $n = 0, k, m+k, 2m, 2m+k, \ldots$, denote a generic system of each set. The positive integers k and m are restricted by $k < m$ and $\delta > 1-k$. Although the orthogonality interval of these polynomials is real, their zeros are generally complex. Here the sum rules $y_r = \sum x_{i,n}^r$, $r = 1, 2, \ldots$, for the zeros $\{x_{i,n}; i = 1, 2, \ldots, n\}$ of the nth-degree polynomials of these sets are studied. It is found that all these quantities vanish except for $r = pm$, p being an arbitrary positive integer. Simple recurrent expressions for y_{pm} are given.

1. Introduction

Recently it has been found (Smith 1982a, b) that there exists an infinite number of new systems of orthogonal polynomials as solutions of non-linear differential equations, some of them playing a relevant role in non-linear physics. Smith (1982a) discovered a new system of orthogonal polynomials in the study of the dispersion of a buoyant contaminant in a fluid and he subsequently showed that there exists an abundance of new non-classical systems of orthogonal polynomials associated with non-linear problems. Indeed, he found (Smith 1982b) that the ordinary differential equations

$$(1-x^m)x\frac{d^2P}{dx^2}-(k-1+\delta x^m)\frac{dP}{dx}+n(n+\delta-1)x^{m-1}P=0 \tag{1}$$

$$x\frac{d^2Q}{dx^2}-(k-1+mx^m)\frac{dQ}{dx}+mnx^{m-1}Q=0 \tag{2}$$

with the positive integers k and m subject to the conditions

$$k<m \qquad \delta>1-k \tag{3}$$

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have orthogonal polynomial solutions, $\{P_n(x; m, k, \delta)\}$ and $\{Q_n(x; m, k)\}$ respectively, with the degrees n restricted to the values

$$n = 0, k, m, m+k, 2m, 2m+k, \ldots. \tag{4}$$

For $m = 1$ and 2, the solutions of the two classes of differential equations are the Jacobi and Laguerre polynomials in the first case and the Gegenbauer and Hermite polynomials in the second. For larger values of m and with the additional condition that k and m are mutually prime, each specification (m, k, δ) in the set of systems $\{P_n(x; m, k, \delta)\}$ and (m, k) in the set of systems $\{Q_n(x; m, k)\}$ defines a new family of orthogonal polynomials. The number of new systems of orthogonal polynomials is (Smith 1982b) uncountably infinite in the P set, and countably infinite in the Q set. This demonstrates that the orthogonal polynomials are not so scarce but, on the contrary, are abundant.

The new systems of polynomials are orthogonal (Smith 1982b) with respect to a weight function $w(x)$ with support $[-1, +1]$ for the P set and $(-\infty, +\infty)$ for the Q set. There are several aspects which make the study of the spectral properties of these polynomials very interesting from both mathematical and physical points of view. Each member of both sets is an orthogonal system but it cannot form a complete basis in L^2_w, owing essentially to the restrictions (4) of the degree n. Because of this, although its interval of orthogonality is real, the zeros of these polynomials belong to the complex plane (Szegö 1975).

Recently, Hendriksen and van Rossum (1987) gave an electrostatic interpretation of the zeros of the systems of the P set corresponding to the specifications $(m, 1, 2qm)$ with $q > 0$ and of all systems of the Q set. In doing so, they extended the work of Stieltjes (who gave (Szegö 1975) an electrostatic interpretation of the zeros of the classical polynomials, then confining himself to the case of the real line) in considering point charge distributions in the complex plane and they assumed the electrical potential to be logarithmic.

Here we shall study the distribution of zeros of the new polynomials (that is, those with $m \geqslant 3$) by means of the determination of the sum rules for the positive integer powers of the zeros $\{x_{i,n}; i = 1, 2, \ldots, n\}$ of an nth-degree polynomial of the two sets, i.e. by calculating the quantities

$$y_r = \sum_{i=1}^{n} x_{i,n}^r \qquad r = 0, 1, 2, \ldots. \tag{5}$$

One should notice that the y quantities, appropriately normalised, are the moments around the origin of the distribution density of zeros $\rho_n(x) = \Sigma_{i=1}^n \delta(x - x_{i,n})$ of the corresponding nth-degree polynomials.

The structure of this paper is as follows. The main results are collected in § 2 and in § 3 we briefly review a method recently proposed by us to calculate the y quantities of the polynomial solutions of an ordinary differential equation of arbitrary order with polynomial coefficients. That method is then applied in § 4 to prove our main results. Finally, some concluding remarks are given.

2. Main results

Here, the y quantities defined by (5), which fully characterise the distribution of the zeros of an nth-degree polynomial of the P and Q sets, are given in terms of the parameters (k, m, δ) or (k, m) which determine the respective differential equations (1) or (2) satisfied by the polynomial.

The sum rules y_r for the zeros of the polynomial $P_n(x; m, k, \delta)$ are equal to zero unless r is a multiple of m. For the latter case, we have found that

$$y_m = \frac{n(n-1)}{2n+\delta-m-1} \tag{6a}$$

$$y_{2m} = \frac{1}{2n+\delta-2m-1}[(2n-m-k)y_m - y_m^2] \tag{6b}$$

$$y_{(p+1)m} = \frac{1}{2n+\delta-(p+1)m-1} \times\left((2n-k-pm)y_{pm} + \sum_{t=1}^{p-1} y_{(p-t)m}y_{tm} - \sum_{t=1}^{p} y_{(p+1-t)m}y_{tm}\right) \tag{6c}$$

where $p = 2, 3, 4, \ldots$.

For the polynomial $Q_n(x; m, k)$, the sum rules y_r vanish except for r equal to a multiple of m. The non-vanishing sum rules have the values given by the following recursion relation:

$$y_m = nm^{-1}(n-k) \tag{7a}$$

$$y_{2m} = (2n-m-k)m^{-1}y_m \tag{7b}$$

$$y_{(p+1)m} = m^{-1}\left((2n-k-pm-1)y_{pm} + \sum_{t=1}^{p-1} y_{(p-t)m}y_{tm}\right) \tag{7c}$$

where $p = 2, 3, 4, \ldots$.

3. Method

In this section we shall briefly describe a method (Case 1980a, b, Dehesa *et al* 1985, Buendia *et al* 1985, 1987) which allows the determination of the sum rules y_r for the zeros of a polynomial of Nth degree satisfying an ordinary differential equation of the form

$$\sum_{i=0}^{h} g_i(x)P_N^{(i)}(x) = 0 \tag{8}$$

where $P_N^{(i)}(x)$ denotes the ith derivative of the Nth-degree polynomial $P_N(x)$ and $g_i(x)$ is the polynomial of degree c_i defined by

$$g_i(x) = \sum_{j=0}^{c_i} a_j^{(i)}x^j. \tag{9}$$

Assuming that all the zeros $\{x_i, i = 1, 2, \ldots, N\}$ of $P_N(x)$ are simple, then the new sum rules $J_s^{(i)}$ defined by

$$J_s^{(i)} = \sum_{\neq} \frac{x_{l_1}^s}{(x_{l_1}-x_{l_2})(x_{l_1}-x_{l_3})\ldots(x_{l_1}-x_{l_i})} \tag{10}$$

(where $\Sigma_{\neq}$ means to sum over all l subject to none of them being equal) verify the relations (Buendia *et al* 1987)

$$\sum_{i=2}^{h} i \sum_{j=0}^{c_i} a_j^{(i)} J_{r+j}^{(i)} = -\sum_{j=0}^{c_1} a_j^{(1)} y_{r+j} \qquad r=0,1,\dots \tag{11}$$

or equivalently, since (Case 1980a) $J_s^{(i)}=0$ for $0 \leqslant s \leqslant i-2$,

$$\sum_{i=2}^{h} i \sum_{m=-1}^{r+c_i-i-1} a_{i+m+1-r}^{(i)} J_{i+m}^{(i)} = -\sum_{j=0}^{c_1} a_j^{(1)} y_{r+j-1} \qquad r=1,2,\dots. \tag{12}$$

The $J_r^{(i)}$ quantities can be expressed (Buendia and Dehesa 1987) for any r and i in terms of the sum rules y_t, $t \leqslant r-1+i$. These expressions are given explicitly for any r and $i=2$ (Case 1980a), $i=3$, 4, 5 (Dehesa *et al* 1985, Buendia *et al* 1985). Also formulae are known for $J_{i+m}^{(i)}$ in terms of the y quantities for any i and $m=-1,0,+1$ (Case 1980a), $m=2,3$ (Buendia *et al* 1985).

The left-hand side of (11) or (12) involves sum rules y_s with $0 \leqslant s \leqslant r+q$ where

$$q = \max_i \{c_i - i;\ i=2,3,\dots,h\} \tag{13}$$

and the right-hand side has quantities y_s with $r-1 \leqslant s \leqslant r+c_1-1$. Therefore, the basic relations (11) or (12) allow us to evaluate the sum rule y_s with $s \geqslant q+1$ recursively in terms of $y_0=N, y_1, y_2, \dots, y_q$.

To calculate the first q of the y sum rules, we have to use the following alternative method (Case 1980b, Buendia *et al* 1987). The polynomial $P_N(x)$ can always be written as

$$P_N(x) = \text{constant} \times \sum_{k=0}^{N} (-1)^k \alpha_k x^{N-k}$$

normalised so that $\alpha_0 = 1$. The α coefficients are related to the y sum rules of zeros by (Buendia *et al* 1987)

$$\alpha_k = [(-1)^k/k!]\, Y_k(-y_1, -y_2, -2y_3, \dots, -(n-1)!\, y_n). \tag{14a}$$

Here the Y_k symbol denotes the well known Bell polynomials of number theory (Riordan 1958) so that for the first few k values one has

$$\begin{aligned} \alpha_1 &= y_1 \\ \alpha_2 &= (y_1^2 - y_2)/2 \\ \alpha_3 &= (y_1^3 - 3y_1 y_2 + 2y_3)/3! \\ \alpha_4 &= (y_1^4 - 6y_1^2 y_2 + 8y_1 y_3 + 3y_2^2 - 6y_4)/4!. \end{aligned} \tag{14b}$$

On the other hand, one can prove (Buendia *et al* 1987) that for $s>1$ the α coefficients satisfy the recursion relations

$$\alpha_s = -\sum_{m=1}^{s} (-1)^m \alpha_{s-m} \sum_{i=0}^{h} \frac{(N-s+m)!}{(N-s+m-i)!} a_{i+q-m}^{(i)} \left(\sum_{i=0}^{h} \frac{(N-s)!}{(N-s-i)!} a_{i+q}^{(i)} \right)^{-1}. \tag{15}$$

The relations (14) and (15) provide another way of calculating all the y quantities in terms of the coefficients $a_j^{(i)}$ of the differential equation defined by (8) and (9). However, due to the high non-linearity of these relations it is clear that this second way is only useful for calculating the first few sum rules y_r, which is what we were looking for.

In summary, we have described in this section a method for calculating the distribution of zeros of an Nth-degree polynomial satisfying an ordinary differential equation of arbitrary order, which is defined by (8) and (9), via the sum rules y_r defined by (5). This method has two steps: (i) the first q sum rules are evaluated by means of (14) and (15) and (ii) the remaining $N-q$ sum rules are determined by the basic recursion relations (11) or (12).

For $q=0$, let us point out that the first step reduces to letting $y_0=N$; then this method reduces to that of Case (1980a) in the case of $h=2$ (i.e. for second-order differential equations), to that of Dehesa *et al* (1985) for $h=4$ and to that of Buendia *et al* (1985) for h equal to any integer value. The method has been applied to a variety of physically interesting systems of orthogonal polynomials which are solutions of: (i) a second-order differential equation, i.e. the classical orthogonal polynomials (Case 1980a) including the generalised Bessel polynomials (Galvez and Dehesa 1984), (ii) a fourth-order differential equation (Dehesa *et al* 1985) such as the so-called Krall classical orthogonal polynomials (Krall 1981 and references therein) and (iii) a sixth-order differential equation (Buendia *et al* 1985) such as the orthogonal polynomials developed by Littlejohn (1982) and Littlejohn and Krall (1983).

The basic idea of our method for $q>0$ is implicitly given by Case (1980b) in dealing with the zeros of Lamé polynomials for which $q=1$. For the general case $q\geqslant 0$ the method has been fully described by us and applied (Buendia *et al* 1987) to the Heine polynomials ($q=2$), the generalised Hermite polynomials ($q=2$) and the new polynomials of Bessel type recently introduced by Hendriksen (1984) ($q=1$).

4. Proofs

Here we will find the values of relations ($6a$–c) and ($7a$–c) for the y sum rules of zeros of the polynomials $P_n(x;\, m, k, \delta)$ and $Q_n(x;\, m, k)$, respectively, starting from the differential equations (1) and (2), respectively, which are satisfied by these polynomials. The method described in the previous section will be used.

Let us begin with $P_n(x;\, m, k, \delta)$. The differential equation (1) satisfied by this polynomial is of the form of (8) and (9) with $h=2$ and the a coefficients given by

$$\begin{array}{llll} a_1^{(2)}=1 & a_{m+1}^{(2)}=-1 & a_j^{(2)}=0 & \text{for any } j\neq 1, m+1 \\ a_0^{(1)}=-(k-1) & a_m^{(1)}=-\delta & a_j^{(1)}=0 & \text{for any } j\neq 0, m \\ a_{m-1}^{(0)}=n(n+\delta-1) & a_j^{(0)}=0 & \text{for any } j\neq m-1. & \end{array} \tag{16}$$

The parameter q is equal to $m-1$ in this case, according to (13). Therefore, first of all we have to evaluate the first $(m-1)$ y quantities by means of (14) and (15). From the latter equation, we have

$$\alpha_s=-\sum_{t=1}^{s}(-1)^t\alpha_{s-t}\sum_{i=0}^{2}\frac{(n-s-t)!}{(n-s+t-i)!}a_{i+m-t-1}^{(i)}\left(\sum_{i=0}^{2}\frac{(n-s)!}{(n-s-i)!}a_{i+m-1}^{(i)}\right)^{-1}.$$

Taking into account the a values given by (16), it is straightforward to see that, while the denominator does not vanish, the numerator is always equal to zero for any s between 1 and $m-1$. So $\alpha_s=0$ for $s=1,2,\ldots,m-1$. Then, according to (14) one has

$$y_s=0 \qquad \text{for } s=1,2,\ldots,m-1. \tag{17}$$

On the other hand, the basic recursion relation (11) reduces for this case as

$$2(J_{r+1}^{(2)}-J_{r+m+1}^{(2)})=(k-1)y_r+\delta y_{r+m} \qquad r=0,1,\ldots \tag{18}$$

where the J value is known to have the value (Case 1980a, Dehesa *et al* 1985, Buendia *et al* 1985)

$$J_s^{(2)} = \begin{cases} 0 & \text{if } s=0 \\ n(n-1)/2 & \text{if } s=1 \\ (n-1)y_1 & \text{if } s=2 \\ \frac{1}{2}[(2n-s)y_{s-1} + \Sigma_{t=1}^{s-2} y_{s-1-t}y_t] & \text{if } s \geqslant 3. \end{cases} \tag{19}$$

We now substitute values of r into (18) and evaluate using (17) and (19). For $r=0$, (18) becomes

$$n(n-k) - \sum_{t=1}^{m-1} y_{m-t}y_t = (2n+\delta-m-1)y_m$$

and using (17) again we obtain directly the required equation (6a). For $r=1$, we obtain $y_{m+1}=0$ in an analogous way. For $r \geqslant 2$, (18) and (19) lead to

$$(2n+\delta-r-m-1)y_{r+m} = (2n-k-r)y_r + \sum_{t=1}^{r-1} y_{r-t}y_t - \sum_{t=1}^{r+m-1} y_{r+m-t}y_t. \tag{20}$$

In the particular cases $r=2, 3, \ldots, m-1$, this expression gives

$$y_s = 0 \qquad \text{for } s = m+2, m+3, \ldots, 2m-1$$

since the two summations are zero due to (17). Also, for $r=m$ the same expression produces the required equation (6b). In general, for $r = pm+p'$ with any positive integer p and $p' = 1, 2, \ldots, m-1$, (20) leads to

$$y_s = 0 \qquad \text{with } s = (p+1)m+1, \ldots, (p+2)m-1$$

and for $r = pm$, with $p = 2, 3, 4, \ldots$, (20) reduces to the recursion relation (6c) which we were seeking. Therefore, the only non-vanishing sum rules y_s are those with order $s = m, 2m, 3m, \ldots$, whose values are given by relations (6a–c) directly in terms of the parameters (m, k, δ) characterising the differential equation (1) with the restrictions (3) which is satisfied by the polynomial $P_n(x;\, m, k, \delta)$ with n subject to the condition (4).

Let us now consider the polynomial $Q_n(x;\, m, k)$. We know that it satisfies the differential equation (2) which can be put into the form of (8) and (9) with $h=2$ and with the a coefficients given by

$$\begin{array}{llll} a_1^{(2)} = 1 & a_j^{(2)} = 0 & & \text{if } j \neq 1 \\ a_0^{(1)} = -(k-1) & a_m^{(1)} = -m & a_j^{(1)} = 0 & \text{if } j \neq 0, m \\ a_{m-1}^{(0)} = mn & a_j^{(0)} = 0 & & \text{if } j \neq m-1. \end{array}$$

Here the q parameter defined by (13) is also $m-1$. Applying the method of § 3 to the Q set leads, in a fully analogous way to the P set, to $y_s = 0$ for $s = 1, 2, \ldots, m-1$ and to the basic relation

$$2J_{r+1}^{(2)} = (k-1)y_r + my_{r+m}$$

for $r = 0, 1, 2, \ldots$. This relation, together with the $J_s^{(2)}$ values given by (19), allows one to obtain all the y quantities from the value $y_0 = n$. Operating as in the P set case, one can easily prove that all the sum rules vanish except those of order $m, 2m, 3m, \ldots$, whose values are indeed given by equations (7a–c).

5. Concluding remarks

We have calculated the power sum rules for the zeros of the polynomials which belong to two infinite sets of families of orthogonal polynomials which are solutions of known non-linear differential equations. The novelty of these polynomials is that, although their interval of orthogonality is real, their zeros belong to the complex plane.

Orthogonal polynomials have been of interest in previous studies of non-linear phenomena, e.g. in connection with the use of the Galerkin method. However, such uses in the past have usually been associated with linearisation of non-linear problems, whereas Smith's work has shown that they can also be useful in direct studies of non-linear phenomena.

It is interesting to remark that all the sum rules y_r vanish except those of an order r which is a multiple of the parameter m characterising the degree of the polynomial coefficients of the differential equation satisfied by the orthogonal polynomial under consideration.

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